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A Classical and a Fuzzy Approach to Study the (m, n)-antiideals of a Semigroup

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The aim of this paper is to characterize a semigroup through its left (right) (m, n)-antiideals. To achieve this, we extend the concept of left (right) antiideals to left (right) (m, n)-antiideals of a semigroup. We provide examples to illustrate these generalized antiideals and investigate their key properties. Additionally, we explore the fuzzification of this new concept from a theoretical perspective, establishing a relationship between the left (right) (m, n)-antiideals of a semigroup and their fuzzy counterparts.

Keywords: Antiideal, fuzzy antiideal, (m, n)-antiideal, fuzzy (m, n)-antiideal

1 INTRODUCTION

Semigroups serve as a fundamental algebraic tool in the analysis of regular languages and finite automata. As abstractions of many real-life situations, the study of semigroup theory has gained significant importance. Like other abstract structures, semigroups are investigated through their subsets, including ideals, filters, mutants, antiideals and their generalizations [2, 3, 7, 8, 10], [11]- [14], [18]. When analyzing the properties and behaviour of a semigroup, dealing with the entire structure can be complex and challenging. Thus, focusing on specific substructures or subsets, such as ideals, filters, or antiideals, allows for a more manageable and targeted mathematical study of the semigroup. This approach simplifies the analysis and helps in understanding the overall structure and properties of the semigroup. The substructures of the semigroup have been defined in order to better model different processes, aspects of the real life. For example, the mutation process, known in biology as a permanent alteration of a genetic property, was algebraically described by Mullin [15] by the help of the mutant notion, that has some kind of mutually exclusive relation with the idempotents (recall that an element a of a semigroup $(S, \cdot)$ is called idempotent if $a \cdot a = a$). A non-empty subset A of a semigroup S is called a mutant, if $A^2 \subset S \setminus A$. In other words, combining two elements of a set leads to a new element that does not satisfy the properties of the initial set, so it belongs to the complement $S \setminus A$. It is clear then that a mutant cannot contain idempotents. S. Lajos was the first one who generalized ideals in semigroups, by defining the (m, n)-ideals of a semigroup. Starting from the definition of a left ideal, known as a non-empty subset A of a semigroup S satisfying $S \cdot A \subset A$, he called a non-empty subset A of a semigroup S to be $(m, 0)$-ideal if $A^m \cdot S \subset A$, for some natural number m. Similarly, A is a $(0, n)$-ideal of S if $S \cdot A^n \subset A$, for some $n \in \mathbb{N}$ and A is called an (m, n)-ideal of a semigroup S if $A^m \cdot S \cdot A^n \subset A$, for some natural numbers m and n. Another (m, n)-substructure of a semigroup is the one of (m, n)-mutant [7]. It generalizes the notion of the mutant of a semigroup. A non-empty subset A is an (m, n)-mutant of a semigroup S if $A^m \subset S \setminus A^n$. Interesting properties and characterizations of semigroups have been obtained by using the so called “anti”-substructures, in particular those of antimutant and antiideal. But their generalizations to the (m, n)-case are not similar. Immediately after the introduction of the concept of (m, n)-ideal, Iseki [8] defined (m, n)-antiideals in semigroups. Thus, a non-empty subset A of a semigroup S is a left (m, n)-antiideal of S if $S \cdot A^m \cap A^n = \emptyset$ (equivalently, $S \cdot A^m \subset S \setminus A^n$); it is a right (m, n)-antiideal of S if $A^m \cdot S \cap A^n = \emptyset$ (equivalently, $A^m \cdot S \subset S \setminus A^n$), while it is called an (m, n)-antiideal if it is a left and a right (m, n)-antiideal of S. Only 20 years after the introduction of the concept of (m, n)-mutant, Dudek [6] defined the (m, n)-antimutants in a very intuitive way. Indeed, by an (m, n)-antimutant A of a semigroup S he meant a non-empty subset A of S such that $S \setminus A^m \subset A^n$. More recently, the theory of (m, n)-ideals has been applied also to other algebraic structures, as for example ordered semigroups [4], LA-semigroups [1] or ordered semihyperrings [16].

However, in many cases, these subsets involve some degrees of vagueness or uncertainty. By grading the substructures, so by introducing the fuzzy ideals, fuzzy mutants, of fuzzy antiideals, the real process can be better algebraically described. Unlike classical sets, where an element either belongs or does not belong to a set, fuzzy sets allow for degrees of membership, reflecting the ambiguity inherent in many real-world situations. Fuzzy algebraic structures, introduced by Rosenfeld [17], extend classical algebraic structures to accommodate uncertainty and imprecision. Since then, they have found applications in diverse fields, including control systems, artificial intelligence, and decision-making, revolutionizing our ability to handle ambiguity and approximate solutions in complex systems. Research on fuzzy algebraic structures continues to evolve, enhancing their adaptability to tackle increasingly intricate real-world problems. Related work can be found in [3, 5, 17].

In [18], Schwarz introduced the concept of antiideals of a semigroup, which was further studied by Iseki [7, 8]. Inspired by recent work and the concept of left (right) (m, n)-antiideal of a semigroup [8], this paper further explores the left (right) (m, n)-antiideal of a semigroup and its fuzzification. The remainder of this paper is organized as follows. Section 2 presents significant examples of left (right) (m, n)-antiideals of a semigroup and related results. Section 3 introduces the fuzzy left (right) (m, n)-antiideal of a semigroup, illustrates it through examples, and examines its key properties. Furthermore, a relationship between the left (right) (m, n)-antiideal of a semigroup S and the fuzzy left (right) (m, n)-antiideal of S is established using the t-level sets.

2 LEFT(RIGHT) (m, n)-ANTIIDEALS OF A SEMIGROUP

This section is dedicated to the classical study of the left(right) (m, n)-antiideals of a semigroup, defined for the first time in [8]. After reviewing their definition and basic properties, supported by several examples, we prove that (m, n)-antiideals don't exist in a group, while an (m, n)-antiideal of a semigroup is never a subsemigroup. Besides, properties related to intersection, union and cartesian product of (m, n)-antiideals are discussed.

Semigroups are abstract algebraic structures that play a fundamental role in various mathematical and technical disciplines. They consist of a non-empty set S endowed with an associative binary operation, usually denoted by "$\cdot$". (If there is no confusion, the operation is omitted). For example, the set of all two by two matrices with real entries under the usual multiplication of matrices is a semigroup and similarly, the set of non-negative rational numbers under the standard addition forms a semigroup, too. A monoid is a semigroup with an identity.

Definition 2.1 [8]. *Let $(S, \cdot)$ be a semigroup and A a non-empty subset of S. Then*

1. *A is called a left antiideal of S if $S \cdot A \cap A = \emptyset$.*
2. *A is a called right antiideal of S if $A \cdot S \cap A = \emptyset$.*
3. *A is an antiideal of S if it is both a left and a right antiideal of S.*

For a better understanding of this notion, we present an example of an antiideal from a real-life perspective.

Example 1. *Let S be the semigroup of all words (not necessary having a sense) under the binary operation "word concatenation", denoted by "$\cdot$". It is clear that the operation is binary closed and associative.*

Consider the subset $A = \{x \in S \mid x \text{ starts with "un" and contains only one "un"}\}$. *For example, the word $understanding$ belongs to the subset A, while the word $ununiform$ does not. Then A is a left antiideal of S, as every element in $S \cdot A$ is either containing more than one "un" or does not start with the letters "un", so the intersection between $S \cdot A$ and A is empty.*

Example 2. *Let $(P_2, \cdot)$ be the semigroup of all positive integers greater than* 1 *under the standard multiplication of integers and, for every $i \in P_2$, consider the singletons $M_i = \{i\}$. Then, for every $i \in P_2$, each set M_i is an antiideal of P_2. This is clear because*

$$(M_i \cdot P_2) \cap M_i = \{2i, 3i, 4i, \ldots\} \cap \{i\} = \emptyset.$$

Definition 2.2. *[8] Let $(S, \cdot)$ be a semigroup, m, n two positive integers, and A a non-empty subset of S. Then*

1. *A is called a left (m, n)-antiideal of S if $S \cdot A^m \cap A^n = \emptyset$.*
2. *A is called a right (m, n)-antiideal of S if $A^m \cdot S \cap A^n = \emptyset$.*
3. *A is called an (m, n)-antiideal of S if it is both a left and a right (m, n)-antiideal of S.*

Clearly, one notice that any $(1, 1)$-antiideal is an antiideal.

In the following example we will notice that (m, n)-antiideals are not just abstract mathematical structures, but they appear also in our real life.

Example 3. *A woman adopted two cats; one has different eyes color and the other one has eyes of the same color. The result of meeting these two cats with two other cats of similar situation led to giving birth to cats with the same properties. By presenting different eye color by "d" and same eye color by "n", the example is illustrated in Table 1.*

One can easily see that $\{d\}$ is an (m, n)-antiideal of C for all positive integers m, n.

Example 4. *Let S denote the set of all possible Lego structures (formed with one block or more blocks) and "$\star$" be the operation translating the Lego combination. One can easily see that $(S, \star)$ is a semigroup, because by combining two Lego structures, one creates a valid Lego structure, while for any three*

$\otimes$	d	n
d	n	n
n	n	n

TABLE 1
The semigroup $(C, \otimes)$

Lego structures, called A, B, and C for simplicity, combining first the Lego structures A and B, and then combining the result with the Lego structure C, is equivalent to combining the Lego structure A with the result of the combination between the Lego structures B and C. This means that the operation "$\star$" is associative.

Let I be the set of all Lego structures containing only one block. Then I is an antiideal of S. Moreover, one notices that J, the set of all lego structures containing one or two blocks, is a $(2, 1)$-antiideal of S.

Example 5. *Let S be a non-singleton set with $0 \in S$ and $(S, \cdot)$ be the semigroup defined by $x \cdot y = 0$, for all $x, y \in S$. Then every non-empty subset of S not containing 0 is an (m, n)-antiideal of S, for all positive integers m, n.*

Proposition 2.3. *[8] Let $(S, \cdot)$ be a semigroup, m, n two positive integers, and A a left (right) (m, n)-antiideal of S. Then $m \geq n$.*

This implies that there are no left (right) (m, n)-antiideals in any semigroup such that $m < n$.

Proposition 2.4. *[8] Let $(S, \cdot)$ be a semigroup and m a positive integer. If S has a left (right) identity, then S doesn't have left (right) (m, m)-antiideals.*

Remark 1. *The same statement is not valid for $m > n$, as one can see in the following example.*

Example 6. *Let P_0 be the semigroup of all non-negative integers under the standard addition of integers. Then $\{1\}$ is a $(2, 1)$-antiideal of P_0, because $(\{1\} + \{1\} + P_0) \cap \{1\} = \{2, 3, \ldots\} \cap \{1\} = \emptyset$.*

Remark 2. *A left (right) (m, n)-antiideal of a semigroup S is not necessary a left (right) antiideal of S. Furthermore, it is not necessary to be a left (right) (m, m)-antiideal, as one can see in the following example.*

Example 7. *Let $M_3(P_0)$ be the semigroup of all three by three matrices with non-negative real entries under the usual multiplication of matrices*

$\star$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	a	a	a	a

TABLE 2
The semigroup $(S,\star)$

and $A = \left\{\begin{pmatrix}3&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}$. *Proposition 2.4 asserts that A is not a left (right) (m,m)-antiideal of $M_3(P_0)$, for any $m \geq 1$. It is easy to see that A is a right $(3,2)$-antiideal of $M_3(P_0)$, because*

$$\left\{\begin{pmatrix}3&0&0\\0&0&0\\0&0&0\end{pmatrix}^3\cdot\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\right\}\cap\left\{\begin{pmatrix}3&0&0\\0&0&0\\0&0&0\end{pmatrix}^2\right\}=$$

$$\left\{\begin{pmatrix}27a&27b&27c\\0&0&0\\0&0&0\end{pmatrix}\right\}\cap\left\{\begin{pmatrix}9&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}=\emptyset,$$

since $b,c\neq 0$.

Remark 3. *A left (right) antiideal of a semigroup S is not always a left (right) (m,n)-antiideal of S, for any positive integers m,n, as showing in the following example.*

Example 8. *Let $(S,\star)$ be the semigroup defined by Table 2. One can easily check that $\{d\}$ is a right antiideal of S, because $\{d\}\star S\cap\{d\}=\{a\}\cap\{d\}=\emptyset$. Moreover, $\{d\}$ is not a right $(2,2)$-antiideal of S, since $\{d\}^2\star S\cap\{d\}^2=\{a\}\cap\{a\}=\{a\}\neq\emptyset$.*

Proposition 2.5. *A group does not possess left (right) (m,n)-antiideals, for any positive integers m,n.*

Proof. Let A be a left (m,n)-antiideal of an arbitrary group $(G,\cdot)$, for some positive integers m,n. Take an arbitrary element $a\in A$. Then $A^m\cdot G\cap A^n=\emptyset$. Since G is a group, it follows that the equation $a^m\cdot g=a^n$ is solvable in G and that $a^{n-m}\in G$ is a solution. The latter implies that $a^n\in A^m\cdot G\cap A^n\neq\emptyset$, which is a contradiction. □

Proposition 2.6. *A left (right) (m, n)-antiideal of a semigroup $(S, \cdot)$ is not a subsemigroup of S.*

Proof. Let $A \neq \emptyset$ be a left (m, n)-antiideal of S. According with Proposition 2.3, we have $m \geq n$. If we suppose by absurd that A is a subsemigroup of S, this implies that $A^{m+1}, A^n \subseteq A$ and hence, $A^{m+1} \subseteq A^n$. The latter implies that $A^{m+1} = A^{m+1} \cap A^n \subseteq S \cdot A^m \cap A^n$, so $S \cdot A^m \cap A^n \neq \emptyset$, which is a contradiction. □

Proposition 2.7. *Let $(S, \cdot)$ be a semigroup and A, B non-empty subsets of S. If A or B is a left (right) (m, n)-antiideal of S, then $A \cap B$ is a left (right) (m, n)-antiideal of S.*

Proof. The proof is straightforward. □

Remark 4. *The union of left (right) (m, n)-antiideals of a semigroup S is not necessary a left (right) (m, n)-antiideal of S, as illustrated in Example 9.*

Example 9. *Let $M_3(P_0)$ be the semigroup defined in Example 7 and* $A_1 = \left\{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$, $A_2 = \left\{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$. *One can easily see that A_1, A_2 are both right $(2, 1)$-antiideals of $M_3(P_0)$. Since*

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in (A_1 \cup A_2)^2 M_3(P_0) \cap (A_1 \cup A_2),$$

it follows that $A_1 \cup A_2$ is not a right $(2, 1)$-antiideal of $M_3(P_0)$.

Theorem 2.8. *Let S_1, S_2 be two semigroups, m, n two positive integers, and A_1 a non-empty subset of S_1, while A_2 is a non-empty subset of S_2. If A_1 is a left (right) (m, n)-antiideal of S_1 or A_2 is a left (right) (m, n)-antiideal of S_2, then $A_1 \times A_2$ is a left (right) (m, n)-antiideal of the direct product $S_1 \times S_2$.*

Proof. The proof is straightforward. □

3 FUZZY LEFT(RIGHT) (m, n)-ANTIIDEALS OF A SEMIGROUP

The aim of this section is to extend to the fuzzy case the concept of (m, n)-antiiideal of a semigroup and study its main properties. More precisely, we define fuzzy left (right) (m, n)-antiideals of a semigroup.

First we will recall some basics elements on the theory of fuzzy sets and fix the notation.

Let X be a universal set and I the real closed interval $[0,1]$. A fuzzy subset A of X is given as an object $A = \{(x, \mu_A(x)) \mid x \in X\}$, where $\mu_A : X \to I$ and $\mu_A(x)$ denotes the membership degree of the element x in X. For simplicity, the fuzzy subset A of X will be identified by its membership degree μ_A. Besides, the t-level set of a fuzzy subset μ of X is the crisp set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$, with $t > 0$. For two fuzzy subsets μ_1, μ_2 of X, a new fuzzy subset $\mu_1 \wedge \mu_2$ is defined as follows:

$$(\mu_1 \wedge \mu_2)(x) = \mu_1(x) \wedge \mu_2(x), \quad \text{for all } x \in X.$$

Definition 3.1. *Let $(S, \cdot)$ be a semigroup and $\mu : S \to [0,1]$ be a non-zero fuzzy subset of S. Then*

1. *μ is called a fuzzy left antiideal of S if $\mu(r \cdot a) \wedge \mu(a) = 0$ for all $r, a \in S$.*
2. *μ is called a fuzzy right antiideal of S if $\mu(a \cdot r) \wedge \mu(a) = 0$ for all $r, a \in S$.*
3. *μ is called a fuzzy antiideal of S if μ is a fuzzy left antiideal of S and a fuzzy right antiideal of S.*

Definition 3.2. *[5] Let $(S, \cdot)$ be a semigroup, $n \geq 2$ a positive integer, and μ a fuzzy subset of S. Then we define*

$$\mu^n(z) = \sup_{x_1 \cdots x_n = z} (\mu(x_1) \wedge \ldots \wedge \mu(x_n)), \quad \text{for } x_1, \ldots, x_n \in S.$$

Definition 3.3. *Let $(S, \cdot)$ be a semigroup, m, n two positive integers, and $\mu : S \to [0,1]$ be a non-zero fuzzy subset of S. Then*

1. *μ is called a fuzzy left (m,n)-antiideal of S if $\mu^n(r \cdot x_1 \cdots x_m) \wedge \mu(x_1) \wedge \ldots \wedge \mu(x_m) = 0$, for all $r, x_1, \ldots, x_m \in S$;*
2. *μ is called a fuzzy right (m,n)-antiideal of S if $\mu^n(x_1 \cdots x_m \cdot r) \wedge \mu(x_1) \wedge \ldots \wedge \mu(x_m) = 0$, for all $r, x_1, \ldots, x_m \in S$;*
3. *μ is called a fuzzy (m,n)-antiideal of S if μ is a fuzzy left and right (m,n)-antiideal of S.*

Proposition 3.4. *Let $(S, \cdot)$ be a semigroup and μ_1, μ_2 two non-zero fuzzy subsets of S. If μ_1 or μ_2 is a fuzzy left (right) (m,n)-antiideal of S, then so $\mu_1 \wedge \mu_2$ is.*

Proof. The proof is straightforward. □

The next result states that the same relationship between m and n exists, as in the classical case, for an (m,n)-antiideal of a semigroup.

Proposition 3.5. *Let $(S,\cdot)$ be a semigroup, m,n two positive integers, and μ be a fuzzy left (right) (m,n)-antiideal of S. Then $m \geq n$.*

Proof.
Suppose that $m < n$. Then there exists an integer $k \geq 1$ such that $n = m+k$. Since μ is a non-zero fuzzy subset of S, it follows that there exists $x \in S$, with $\mu(x) \neq 0$. Being μ a fuzzy left (m,n)-antiideal of S, it follows that $\mu^{m+k}(x^k \cdot x^m) \wedge \mu(x) = 0$, where $\mu(x) \neq 0$. This implies that $\mu^{m+k}(x^k \cdot x^m) = 0$. Since $\mu^{m+k}(x^k \cdot x^m) \geq \mu(x)$, it follows that $\mu(x) \leq 0$, that is a contradiction. □

Example 10. *Let $M_3(P_0)$ be the semigroup defined in Example 7 and define the fuzzy subset μ of $M_3(P_0)$ as follows:*

$$\mu(B) = \begin{cases} 0.7, & \text{if } B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\ 0.6, & \text{if } B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\ 0 & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy right $(3,2)$-antiideal of $M_3(P_0)$, but it is not a fuzzy right antiideal of $M_3(P_0)$. This is clear because

$$\mu\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \wedge \mu\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 0.7 \neq 0.$$

Theorem 3.6. *Let $(S,\cdot)$ be a semigroup, m,n two positive integers, and A a non-empty subset of S. Then A is a left (right) (m,n)-antiideal of S if and only if μ_A is a fuzzy left (right) (m,n)-antiideal of S, where $\mu_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$*

Proof. We prove the statement for the left case, similarly it holds for the right one. Let A be a left (m,n)-antiideal of S and $x_1,\ldots,x_m,r$ arbitrary elements in S. If $x_i \notin A$ for some $i \in \{1,\ldots,m\}$, then $\mu_A^n(r \cdot x_1 \cdots x_m) \wedge \mu_A(x_1) \wedge \ldots \wedge \mu_A(x_m) = 0$. Otherwise, if $x_i \in A$ for all $i \in \{1,\ldots,m\}$, then $r \cdot x_1 \cdots x_m \notin A^n$ because $S \cdot A^m \cap A^n$ is an empty set. Hence, there exist no $z_1,\ldots,z_n \in A$ with $r \cdot x_1 \cdots x_m = z_1 \cdots z_n$. The latter equality implies that $\mu_A^n(r \cdot x_1 \cdots x_m) = 0$. Therefore, $\mu_A^n(r \cdot x_1 \cdots x_m) \wedge \mu_A(x_1) \wedge \ldots \wedge \mu_A(x_m) = 0$.

Conversely, let μ_A be a fuzzy left (m,n)-antiideal of S and α be an arbitrary element in $S \cdot A^m \cap A^n$. Then there exist $y_1, \ldots, y_m, a_1, \ldots, a_n \in A$ and $r \in S$ such that $\alpha = r \cdot y_1 \cdots y_m = a_1 \cdots a_n$. Having $\mu_A(a_i) = \mu_A(y_j) = 1$ for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, it follows that $\mu_A^n(r \cdot y_1 \cdots y_m) \wedge \mu_A(y_1) \wedge \ldots \wedge \mu_A(y_m) = 1 \neq 0$ and the proof is now complete. □

Theorem 3.7. *Let $(S, \cdot)$ be a semigroup, m, n two positive integers, and μ a fuzzy subset of S. Then μ is a fuzzy left (right) (m,n)-antiideal of S if and only if each t-level set $\mu_t \neq \emptyset$ is a left (right) (m,n)-antiideal of S.*

Proof. Let $\mu_t \neq \emptyset$ be a left (m,n)-antiideal of S, $x_1, \ldots, x_m, r$ arbitrary elements in S satisfying the condition $\mu^n(r \cdot x_1 \ldots x_m) \wedge \mu(x_1) \wedge \ldots \wedge \mu(x_m) = t > 0$. Then each $x_i \in \mu_t \neq \emptyset$ and hence, $r \cdot x_1 \cdots x_m \in R \cdot \mu_t^m$. Since

$$\mu^n(r \cdot x_1 \cdots x_m) = \sup_{z_1 \cdots z_n = r \cdot x_1 \cdots x_m} \mu(z_1) \wedge \ldots \wedge \mu(z_n) \geq t,$$

it follows that there exist $y_1, \ldots, y_n \in S$ such that $r \cdot x_1 \cdots x_m = y_1 \cdots y_n$ and $\mu(y_1) \wedge \ldots \wedge \mu(y_n) \geq t$. The latter implies that $r \cdot x_1 \cdots x_m = y_1 \cdots y_n \in \mu_t^n$ and hence, $r \cdot x_1 \cdots x_m = y_1 \cdots y_n \in \mu_t^m \cap S \cdot \mu_t^n = \emptyset$.

Conversely, let μ be a fuzzy left (m,n)-antiideal of S and α an arbitrary element in $S \cdot \mu_t^m \cap \mu_t^n$. Then there exist $x_1, \ldots, x_m, z_1, \ldots, z_n \in \mu_t, r \in S$ such that $\alpha = r \cdot x_1 \cdots x_m = z_1 \cdots z_n$. This implies that $\mu^n(r \cdot x_1 \cdots x_m) \geq \mu(z_1) \wedge \ldots \wedge \mu(z_n) \geq t$ and hence, $\mu^n(r \cdot x_1 \cdots \mu(x_m)) \wedge \mu(x_1) \wedge \ldots \wedge \mu(x_n) \geq t \neq 0$. Similarly, one proves the statement for (fuzzy) right (m,n)-antiideals. □

Theorem 3.8. *Let $(S, \cdot)$ be a semigroup. Then every left (right) (m,n)-antiideal of S is a t-level set of a fuzzy left (right) (m,n)-antiideal of S.*

Proof. Let A be a left (m,n)-antiideal of S and define the fuzzy subset μ on S as follows:

$$\mu(x) = \begin{cases} 0.65, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

One can easily see that μ is a fuzzy left (m,n)-antiideal of S and that $A = \mu_{0.65}$. □

Remark 5. *A left (right) (m,n)-antiideal of a semigroup S could be considered as a level set of different fuzzy left (right) (m,n)-antiideals of S, as clearly illustrated in Example 11.*

Example 11. *We continue with Example 2. Let* $\{2\}$ *be the* $(2,1)$*-antiideal of* P_2 *and* μ_1, μ_2 *be two fuzzy subsets of* P_2 *described as follows:*

$$\mu_1(x) = \begin{cases} 0.9, & \text{if } x = 2; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } \mu_2(x) = \begin{cases} 0.8, & \text{if } x = 2; \\ 0.7, & \text{if } x = 3; \\ 0, & \text{otherwise.} \end{cases}$$

One can easily see that μ_1, μ_2 *are two different fuzzy* $(2,1)$*-antiideals of* P_2 *and* $\{2\} = (\mu_1)_{0.9} = (\mu_2)_{0.8}$.

4 CONCLUSIONS

This paper has dealt with left (right) (m,n)-antiideals of a semigroup and their fuzzification from a theoretical approach. Examples and fundamental properties of left (right) (m,n)-antiideals of a semigroup were presented. Besides, through the level sets, a relationship between these concepts and their fuzzy counterparts was established. The interplay between classical and fuzzy algebraic structures opens doors for further research, offering insights into more complex algebraic systems and their real-world applications. In particular, the relationship between (fuzzy) (m,n)-antiideals and (fuzzy) (m,n)-antimutants deserves to be further more investigated.

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